A THEOREM ON LATTICE-POINTS.*

By Aubrey J. Kempner.

Since lattice-point systems are of importance in many branches of mathematics,† there is justification for studying them for their own sake. The problem treated in this paper is also of interest on account of the very elementary methods required and because its complete solution is given by a pointwise discontinuous function, whereas generally such functions are artificially built up in order to illustrate certain peculiarities that real functions may possess.

In a rectangular system of coördinates we consider the lattice-point system x_i, y_j; $x_i = 0, \pm 1, \pm 2, \cdots, y_i = 0, \pm 1, \pm 2, \cdots$. Our problem shall consist of discussing the system of rectilinear paths of finite width and extending to infinity in both directions, which may be laid through our point-system, when points may lie on either boundary-line of the path, but not in the path.

We prove first

Theorem I. *Every straight line in our coördinate-plane belongs to one of the four types: (a) lines of rational slope containing no points of the system, (b) lines of rational slope containing an infinite number of our points, (c) lines of irrational slope containing no points, (d) lines of irrational slope containing exactly one point.*

Proof. (a) Lines satisfying (a) obviously exist, for example, $y = \alpha$, or $y = x - \alpha$, α any rational number, but not an integer.

(b) is true because a line containing one of our points must necessarily contain a second one when the slope is rational, and a line containing two points (q_1, p_1), (q_2, p_2) contains all points

$$(q_1 + k(q_2 - q_1),\ p_1 + k(p_2 - p_1)), \qquad k = \pm 1, \pm 2, \pm 3, \cdots.$$

Lines parallel to the y-axis are to be counted as of rational slope.

* Presented to the American Mathematical Society, Chicago, April, 1917.

† Such systems are of importance for example in number-theory, in the theory of binary quadratic forms, the theory of algebraic numbers, the theories of elliptic functions and of elliptic modular functions, and, when we consider systems in space, in crystallography. Two of the most valuable studies of plane lattice-point systems are: Klein, Ausgewählte Kapitel der Zahlentheorie, I, II, Hectographierte Vorlesung 1895/6, and Minkowski, Diophantische Approximationen, 1907. Also parts of Minkowski's Geometrie der Zahlen deal with plane lattice-point systems.

In Klein's treatment the lattice-point system is made the basis of the theories of quadratic forms (Pell's equation, etc.), of elliptic functions and of elliptic modular functions.

(*c*) Any line $y = \tan\varphi(x - c)$, $\tan\varphi$ irrational, c rational, but not an integer, does not contain any point of our system.

(*d*) A line of irrational slope clearly does not contain more than one of our points; on the other hand, in case it should contain none, we can draw a line parallel to it through one of our points.

It may be mentioned that Theorem I still holds when we consider the everywhere dense set of points having both coördinates rational; it also holds for the set of points having the algebraic numbers for coördinates, provided we classify our straight lines according to whether their slope has an algebraic or a transcendental value.

THEOREM II.* *Any line of irrational slope has on either side an infinite number of points lying closer to it than any assigned distance.*

Proof. The following theorem is well known and different proofs for it are in existence: "Let α be any real irrational number, then there exist an infinite number of integers p, q such that

$$0 < \left|\frac{p}{q} - \alpha\right| < \frac{1}{q^2}."$$

This theorem states in one respect much more than is needed for the proof of II, while in another respect it does not state quite enough. It will be just as convenient to derive the inequalities which we shall need as to modify the theorem just quoted. We therefore prove first the following

LEMMA. *Let α be any real irrational number, c any real number, and $\epsilon > 0$ arbitrarily small, but fixed. Then there exist four integers p_1, p_2, q_1, q_2 so that*

$$c < q_1\alpha - p_1 < c + \epsilon$$

and

$$c - \epsilon < q_2\alpha - p_2 < c.$$

Proof of lemma. We make use of Minkowski's "zirkulare Anordnung von Intervallen"—circular arrangement of intervals—by means of which he proves the more powerful inequality quoted above (in slightly modified form).†

Consider a circle of radius $1/2\pi$, so that the length of the circumference is unity, and, taking any point O on the perimeter as a starting-point, measure along the circumference the lengths α, 2α, 3α, $\cdots$ in inf., going

* In his book mentioned above, Klein gives a much more precise theorem on the degree of approximation in case the line passes through a lattice-point.—By a remarkable geometrical interpretation, Klein's theorem (see p. 17 of his book) places the theory of approximation to irrational numbers by continued fractions in a new light. Compare also MacMillan, A Theorem connected with irrational numbers, Amer. Journal of Math., 1916, v. 38, p. 387. H. J. S. Smith, Collected Mathematical Papers, vol. II, 1894, p. 146, also mentions the same interpretation.

† See: Diophantische Approximationen, p. 4.

around the circle as often as necessary. No two points thus marked can coincide (for *rational* α we should obtain a finite set of equidistant points along the circumference) and, since there are an infinite number of them, they must have at least one point of condensat on. Therefore there are among our points two whose distance measured along the circumference is smaller than ϵ, say equal to ϵ_1, $0 < \epsilon_1 < \epsilon$, and we can find four integers, m_1, m_2, k_1, k_2, some positive and some negative, so that

$$\begin{aligned} |m_1\alpha - k_1 - (m_2\alpha - k_2)| &= |(m_1 - m_2)\alpha - (k_1 - k_2)| \\ &= |m\alpha - k| = \epsilon_1 < \epsilon, \end{aligned}$$

where m, k are again two integers. Then the points

$$|\rho \cdot m \cdot \alpha - \rho \cdot k| = \rho \cdot \epsilon_1, \qquad \rho = 1, 2, 3, \cdots \text{ in inf.},$$

cover the circumference of our circle in such manner that the distance, measured along the circumference, between any two consecutive points is smaller than ϵ, and the point c, measured off on the circumference from the starting-point O, lies in one of these intervals or in an end-point of an interval. This proves our Lemma.

Theorem II is now easily derived: Let $y = m(x - c) = mx - \gamma$, m irrational, c any real number, be the equation of our line. Without loss of generality, m may be assumed positive, as is easily seen. Then $|mq - p - \gamma| = \nu$ is the vertical distance of any point (q, p) from the line and the perpendicular distance δ is smaller than ν. By our Lemma we can find four integers p_1, p_2, q_1, q_2 so that

$$\gamma < q_1 m - p_1 < \gamma + \epsilon$$

and

$$\gamma - \epsilon < q_2 m - p_2 < \gamma,$$

$\epsilon > 0$ arbitrarily assigned, and therefore ν, and all the more δ, an arbitrarily small positive quantity. By choosing ϵ smaller and smaller, we find as many points on either side of our line as we like.

THEOREM III. *When $\tan \varphi = p/q$, p, q relatively prime, then the broadest possible path in the direction φ, which does not contain any lattice-point in its interior, has the width*

$$d = \frac{1}{\sqrt{p^2 + q^2}}.$$

When $\tan \varphi$ is irrational, no path of finite width exists in the direction φ which does not contain an infinite number of points in the interior.

Proof. For simplicity of statement, we may assume $0 \leqq \varphi \leqq (\pi/2)$, without real loss of generality. For irrational values of $\tan \varphi$ the proof follows immediately from II. For $\varphi = 0$ and for $\varphi = \pi/2$, our paths are of width unity.

We assume $\tan \varphi = p/q$, p, q relatively prime. If one or both of the border-lines should not pass through one of our lattice-points, we should be able to make the path broader by replacing the actual border-lines by parallel ones. We may, without loss of generality, assume that one of the borders of our path contains the origin; then the distance of any point (q_1, p_1) from this border-line is

$$d = \frac{|qp_1 - pq_1|}{\sqrt{p^2 + q^2}}.$$

Since p, q are relatively prime, it is possible to find integers p_1, q_1 such that $|qp_1 - pq_1| = 1$,* and this is the smallest possible value of the numerator (besides 0, which corresponds to points in the line). This proves III.

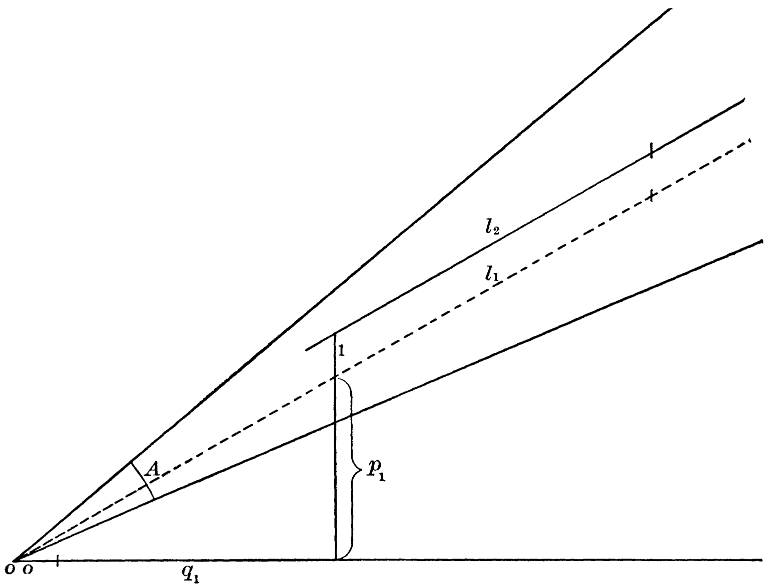

FIG. 1.

THEOREM IV. *Always letting one border (for example the left-hand border) of our paths pass through a fixed lattice-point, and considering only the paths which correspond to a continuous change in direction through any fixed angle, however small, the sum of the widths diverges.*

Proof. Assume the fixed point, through which the left-hand borders of our paths pass, to be the origin, and consider the sector of angle A

* It is easily possible to avoid the diophantine equation and to make the proof entirely elementary. See Theorem V, second proof.

swept over by the left-hand border. We can find in this sector (which we may assume to be in I. Quadrant) at least one line l_1 passing through (0, 0) and a lattice-point (q_1, p_1) (see Fig. 1), and so that also $(q_1, p_1 + 1)$ lies inside of the sector. Then the line l_2 parallel to l_1 and passing through $(q_1, p_1 + 1)$ contains an infinite number of lattice-points $(q_1, p_1 + 1)$, $(2q_1, 2p_1 + 1)$, $(3q_1, 3p_1 + 1)$, $\cdots$ in inf., which all lie in our sector and no two of which are collinear with (0, 0). Consider the set of paths whose left-hand borders pass through (0, 0) and $(\lambda q_1, \lambda p_1 + 1)$, $\lambda = 1, 2, 3, \cdots$. Any pair of numbers λq_1, $\lambda p_1 + 1$ are not necessarily relatively prime, but the effect of using in

$$d = \frac{1}{\sqrt{p^2 + q^2}}$$

for p, q two numbers not relatively prime will be to make our path narrower than necessary. Hence, when we let the left-hand border rotate through a fixed angle, however small, we shall have in

$$\sum_{\lambda=1}^{\infty} \frac{1}{\sqrt{(\lambda q_1)^2 + (\lambda p_1 + 1)^2}}$$

the sum of the widths of only a sub-set of all paths satisfying the conditions of IV, and some terms of the series may be smaller than the width of the corresponding broadest paths. Therefore IV will be proved when we show that

$$\sum_{\lambda=1}^{\infty} \frac{1}{\sqrt{(\lambda q_1)^2 + (\lambda p_1 + 1)^2}}$$

diverges. This follows immediately from

$$\frac{1}{\sqrt{(\lambda q_1)^2 + (\lambda p_1 + 1)^2}} > \frac{1}{\lambda} \cdot \frac{1}{\sqrt{q_1^2 + (p_1 + 1)^2}}.$$

Evidently the sum of the widths still diverges when we consider in any sector only those paths of which the maximum width is smaller than any assigned positive value.

THEOREM V. *Consider a path of maximum width one border of which passes through* (0, 0) *and* (q, p), p, q *relatively prime. The area of the path measured between* (0, 0) *and* (q, p) *is unity.*

This is, as will be seen, exactly the theorem which states that, for doubly-periodic functions (lemniscatic functions corresponding to a *square* lattice-point system), the areas of any two primitive period-parallelograms are equal.

Proof. First proof. The area of the path within the interval considered is equal to a rectangle of sides $\sqrt{p^2 + q^2}$ and $1/\sqrt{p^2 + q^2}$.

Second proof. A geometric proof is indicated by Fig. 2.

We first let the shaded parts of our parallelogram $(abcd)$ slide down to the x-axis, and then let those parts which are not already in the square $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$ slide along the x-axis in the way indicated by the figure.

To show how the proof applies to the general case, assume for p and q (relatively prime) the rectangle $(0, 0)$, $(q, 0)$, $(0, p)$, (q, p), and draw the diagonal from $(0, 0)$ to (q, p). This diagonal intersects each of the lines $y = k$, $(k = p, p - 1, \cdots, 2, 1)$; the distances of these points of intersection from the side of the rectangle whose equation is $x = q$ are, respectively,

$$0, \frac{q}{p}, \frac{2q}{p}, \cdots, \frac{(p-1)q}{p}.$$

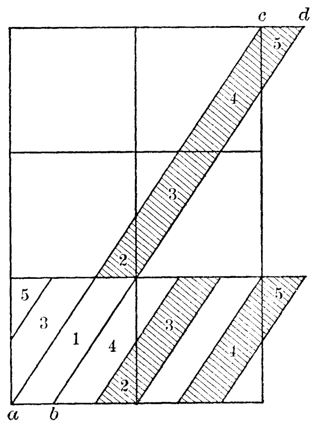

FIG. 2.

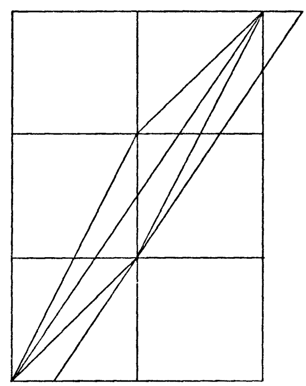

FIG. 3.

Since the numerators of these fractions form a complete residue system modulo p, the horizontal distance of exactly one lattice-point lying in our rectangle to the right of the diagonal must be $1/p$, and the geometrical translations can be carried out as indicated in Fig. 2.

Theorem III can thus be deduced from Fig. 2, when constructed for any given p and q.

Fig. 3 indicates the relation between Theorem V and the theorem on primitive period-parallelograms.

VI. *Discussion of the relation between the width of the path and the angle* φ. We consider all paths of which one border (say the left-hand border) passes through a fixed lattice-point. We may assume this fixed point to be the origin.

The manner in which the width (d) varies with φ will best be described

if we consider d as a function of tan φ; $y = f(x)$, writing y for d and x for tan φ.

The following properties of $f(x)$ follow immediately from the theorems already proved:

1. $f(x) = 0$ for x irrational.
2. $f\left(\frac{p}{q}\right) = \frac{1}{\sqrt{p^2 + q^2}}$ for p, q relatively prime.
3. $f(0) = f(\infty) = 1$.
4. $f\left(-\frac{p}{q}\right) = f\left(\frac{p}{q}\right)$.
5. $f\left(\frac{p}{q}\right) = f\left(\frac{q}{p}\right)$.
6. $0 \leqq f(x) \leqq 1$ for all values of x.

On account of 4 we may restrict our discussion to positive values of x; on account of 5 we may make the further restriction $0 < x \leqq 1$.

We consider rational values of the argument, *always assuming* p, q *relatively prime.* We have

$$f\left(\frac{p}{q}\right) = \frac{1}{q} \cdot \frac{1}{\sqrt{1 + \left(\frac{p}{q}\right)^2}}.$$

The second factor on the right side, considered as a function of p/q, does not present any peculiarity. We therefore first investigate the function $F(p/q) = 1/q$, $F(x) = 0$ for x irrational.*

$F(x)$ clearly possesses properties 1 and 6, and we assume $F(0) = 1$. Besides, we have a new property:

$$7.\ F\left(\frac{p}{q}\right) = F\left(1 - \frac{p}{q}\right).$$

* This function is one of the standard examples of pointwise discontinuous integrable functions. Compare for example: Encycl. d. sciences math., II, 1, 2, Montel, Intégration et dérivation, p. 176. Thomae, Bestimmte Integrale, 1908, p. 19. Hobson, Real variables, 1907, gives this function among the exercises on p. 348 as possessing the following properties: (*a*) pointwise discontinuous; (*b*) the Riemann integral (and therefore also the Lebesgue integral) over the interval $0 \cdots 1$ exists: $\int_0^1 F(x)dx = 0$; (*c*) only a finite number of points in the interval $0 \cdots 1$ at which $F(x)$ exceeds an assigned positive number, however small.

See also Hardy, Pure Mathematics, 1908, p. 184, for a related function.

Since, however, a closer investigation of the structure of the function along the lines indicated by our problem was not intended in any of the treatises just referred to, this investigation is carried out in the text.

From the definition of $F(p/q)$ follows:

(a) $F(1/n') = 1/n'$ for $n' = 1, 2, 3, 4, \cdots$;
(b) $F(2/n'') = 1/n''$ for $n'' = 3, 5, 7, 9, \cdots$, that is, for $n'' > 2$ and n'' relatively prime to 2;
(c) $F(3/n''') = 1/n'''$ for $n''' = 4, 5, 7, 8, \cdots$, that is, for $n''' > 3$ and n''' relatively prime to 3; and in general
(k) $F(k/n^{(k)}) = 1/n^{(k)}$ for all $n^{(k)}$ such that $n^{(k)} > k$ and $n^{(k)}$ relatively prime to k.

All points of $F(x)$ given by (a) lie on the line joining (0, 0) and (1, 1), and they have on this line one point of condensation, at (0, 0). All points given by (b) lie on the line joining (0, 0) and (1, $\frac{1}{2}$), and they likewise have one point of condensation, again at (0, 0). In the same way we see that every line passing through (0, 0) and (1, $1/k$), $k = 1, 2, 3, 4, \cdots$, contains an infinite number of points with an only point of condensation at the origin. On account of property 7, every line joining

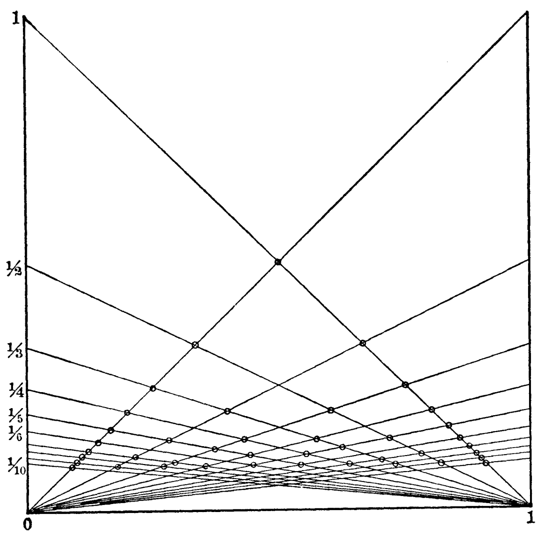

FIG. 4.

(1, 0) and (0, $1/k$) also contains an infinite number of points with an only point of condensation at (1, 0). The points of the locus lying on the second set of lines are the same as the points lying on the first set, and, apart from the everywhere dense set of points lying on the x-axis, all points of our locus are determined as a sub-set of the points of intersection of our two sets of lines. (See Fig. 4.) It is also easy to see how our points are distributed along a set of lines parallel to the x-axis: On

each line $y = 1/n$, $n = 2, 3, 4, 5, \cdots$, we have exactly $\varphi(n)$ points of our locus, where $\varphi(n)$ gives the number of positive integers smaller than n and relatively prime to n, and all points of our locus lie on these lines and on the x-axis (except the two points $f(0) = f(1) = 1$). Hence there are exactly $2 + \sum_{k=2}^{n} \varphi(k)$ points lying on or above the line $y = 1/n$. All except a finite number of points therefore lie in the region $0 < y \leqq \epsilon$, ϵ any positive number. (Compare last footnote.)

It is now easy to pass from $F(x)$ to $f(x)$, since we must only multiply every ordinate of $F(p/q)$ by $1/\sqrt{1 + (p/q)^2}$.

Since $1/\sqrt{1 + x^2}$ is a monotone decreasing function in the interval $0 \cdots 1$, starting with the functional value 1 and ending with $\frac{1}{2}\sqrt{2}$, the effect of the multiplication is obvious.

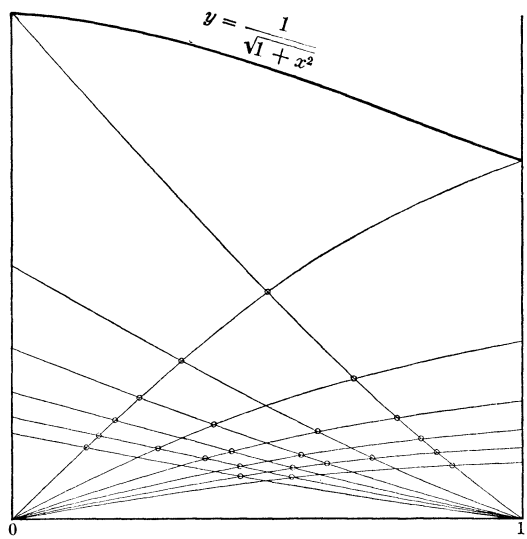

FIG. 5.

The set of straight lines starting from $(0, 0)$ will be transformed into a set of curves through $(0, 0)$, each one of which is concave downward and lies entirely underneath the corresponding straight line; while the set of lines starting from $(1, 0)$ will be transformed into a set of curves each one of which passes through $(1, 0)$ and also through the point of intersection of the corresponding straight line with the y-axis, while for intermediate values of x the (concave upward) curve lies entirely below the line. The points of intersection of our two sets of curves, corresponding to those points of intersection in Fig. 4 which are points of $y = F(x)$, yield points of $y = f(x)$, and exactly all points of $f(x)$ except the everywhere dense

set of irrational points lying on the x-axis (see Fig. 5). Like $F(z)$, our function is pointwise discontinuous, integrable from 0 to 1, with

$$\int_0^1 f(x)dx = 0,$$

and all but a finite number of points lie within any band $0 < y \leqq \epsilon$, ϵ any given positive number, for $0 < x < 1$.

The broadest paths in our system are approximately realized, and are frequently clearly visible, in a well-planted corn field.

Without important modifications, the investigation may be carried over to rectangular and parallelogram arrangements of points, and therefore in particular to the case when our points are vertices of a system of equilateral triangles filling the plane. In generalizing into space-systems, it will be natural to consider, corresponding to the paths in the plane, the space between parallel planes which may be laid through the system so that no lattice-points lie between the planes.

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